Coupling for continuous state branching processes with immigration

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Coupling

- A coupling (X_t, Y_t): if both X_t and Y_t are Markov processes associated with the same transition probability P_t (with different initial distribution μ_1 and μ_2), where X_t and Y_t are called the marginal processes of the coupling.
- A coupling (X_t, Y_t) is called successful if the coupling time

$$T := \inf\{t \geq 0 : X_t = Y_t\} < \infty, \ a.s.$$

Then

$$\|\mu_1 P_t - \mu_2 P_t\|_{var} := \sup_{\|f\| \leq 1} \left|\mathbb{E}[f(X_t)] - \mathbb{E}[f(Y_t)]\right| \leq 2\mathbb{P}(T > t)$$

which goes to 0 as $t \to \infty$.

Coupling

Definition. A strong Markov process with P_t is said to have a coupling property if for any μ_1, μ_2, $\lim_{t\to\infty} \|\mu_1 P_t - \mu_2 P_t\|_{var} = 0$.

The definition is equivalent to one of the following:

- All bounded time-space harmonic functions (i.e. $u(t,\cdot) = P_s u(t+s,\cdot)$) are constant.
- The tail σ-algebra of the process is trivial.

See Cranston and Greven (1995) and Lindvall (1992).

Motivation

A growing literature on coupling for jump processes.

- Basic coupling and refined basic coupling:

  Chen (2004): Q processes; Schilling and Wang (2011): Levy processes; Wang (2012): Ornstein-Uhlenbeck type processes; Luo and Wang (2018): Levy driven SDE; Li and Wang (2020), Li, Li, Wang and Zhou (2022) non-linear branching processes....⇒ exponential ergodicity and ergodicity.

- Coupling by change of measure:

  Wang (2012): Coupling and Applications; Wang (2011):Ornstein-Uhlenbeck type processes; Zhang and Zheng (2018): CB diffusion processes; Huang and Zhao (2019): nonlinear CB diffusion processes...⇒ ergodicity, Harnack inequality, gradient estimate for the processes.

Continuous state branching processes (CB)

- Let $\{\xi_{n,i}\}$ be positive integer-valued i.i.d. random variables. A Galton-Watson branching process $\{Z_n\}$ is defined by

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_{n,i}, \quad n \geq 1.$$

Then ($m := \mathbb{E}[\xi_{1,1}]$)

$$Z_n - Z_{n-1} = (m-1)Z_{n-1} + \sum_{i=1}^{Z_{n-1}} (\xi_{n,i} - \mu) \tag{1}$$

- The similar structure of a typical continuous state branching process is given by

$$dX_t = -bX_t dt + \int_0^{X_{t-}} \int_0^{\infty} \xi \tilde{N}(dt, du, d\xi) \tag{2}$$

where $\tilde{N}(dt, du, d\xi)$ = compensated Poisson random measure on $(0, \infty)^3$. See Bertoin and Le Gall (2006), Dawson and Li (2006).

Stochastic equation for CB processes

Suppose that $\sigma \geq 0$ and b are constants, and $\mu(d\xi)$ is σ-finite Levy measure on $(0, \infty)$ satisfying $\int_0^\infty \xi \wedge \xi^2 \mu(d\xi) < \infty$.

Theorem (Dawson and Li (2006)) There is a pathwise unique positive solution to

$$X_t = X_0 - \int_0^t bX_s ds + \int_0^t \sigma\sqrt{X_s} dB_s + \int_0^t \int_0^{X_{s-}} \int_0^\infty \xi \tilde{N}(ds, du, d\xi)$$

where B_t =Brownian motion; $N(ds, du, d\xi)$ =Poisson random measure with intensity $ds du \mu(d\xi)$

- If we replace X_s by 1, the above equation becomes the Lévy-Itô representation of some Levy process $\{L_t\}$ characterized by

$$\Psi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_0^\infty (e^{-z\xi} - 1 + z\xi)\mu(d\xi).$$

$\{X_t\}$ is called a general CB process with branching mechanism Ψ.

Typical examples of CB processes

- Feller (1951)

$$dX_t = -bX_t dt + \sigma\sqrt{X_t}dB_t$$

X_t is a CB diffusion process with $\Psi(z) = bz + \frac{1}{2}\sigma^2 z^2$.

- Lambert (2007); Fu and Li (2010)

$$dX_t = -bX_t dt + \sigma_z \sqrt[\alpha]{X_{t-}}dZ_t$$

where $\{Z_t\}$ is a spectrally positive α-stable Lévy process with $\alpha \in (1,2)$. In this case, X_t is a CB pure jump process with $\Psi(z) = bz + \sigma_z^\alpha z^\alpha$.

Transition semigroup

The transition semigroup for CB processes X_t with branching mechanism $\Psi(\cdot)$ given by

$$\mathbb{E}_x\left[e^{-pX_t}\right] = \exp\left[-xv(t,p)\right],$$

where $v : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$\frac{\partial v(t,p)}{\partial t} = -\Psi(v(t,p)), \quad v(0,p) = p$$

and Ψ given by

$$\Psi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_0^\infty (e^{-z\xi} - 1 + z\xi)\mu(d\xi).$$

Extinction

- The extinction time of CB is defined by

$$\tau_0 = \inf\{t \geq 0 : X_t = 0\}.$$

- Grey (1974): For subcritical ($b > 0$) or critical ($b = 0$) CB processes,

$$\mathbb{P}(\tau_0 < \infty) = 1 \iff \text{Grey condition holds, i.e.,}$$

there is some constant $\theta > 0$ such that

$$\int_\theta^\infty \Psi(z)^{-1} dz < \infty.$$

Typical examples when Grey's condition holds

- Feller (1951)

$$dX_t = -bX_t dt + \sigma\sqrt{X_t} dB_t$$

X_t is a CB diffusion process with $\Psi(z) = bz + \frac{1}{2}\sigma^2 z^2$.

- Lambert (2007); Fu and Li (2010)

$$dX_t = -bX_t dt + \sigma_z \sqrt[\alpha]{X_{t-}} dZ_t$$

where $\{Z_t\}$ is a spectrally positive α-stable Lévy process with $\alpha \in (1, 2)$. In this case, X_t is a CB pure jump process with $\Psi(z) = bz + \sigma_z^\alpha z^\alpha$.

CB processes with immigration (CBI)

- $\{Y_t\}$ is called a general CBI process with(Ψ, Φ) given by

$$\begin{aligned} Y_t &= Y_0 - \int_0^t bY_s ds + \int_0^t \sigma\sqrt{Y_s} dB_s \\ &\quad + \int_0^t \int_0^{Y_{s-}} \int_0^\infty \xi \tilde{N}(ds, du, d\xi) + S_t, \end{aligned}$$

where S_t is a subordinator with Lévy exponent

$$\Phi(z) = az + \int_0^\infty (1 - e^{-zu}) n(du).$$

- Kawazu and Watanabe (1971): transition semigroup given by

$$\mathbb{E}_x \left[e^{-pX_t} \right] = \exp\left[-xv(t,p) - \int_0^t \Phi\Big(v(s,p)\Big) ds \right],$$

where $v : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$\frac{\partial v(t,p)}{\partial t} = -\Psi(v(t,p)), \quad v(0,p) = p$$

Asymptotic behaviors of CBI processes

Theorem (Pinsky (1972), Li (2010), Foucart, M., and Yuan (2021)). Consider a CBI process $(Y_t, t \geq 0)$.

	$\int_0 \frac{\Phi(u)}{\lvert\Psi(u)\rvert} du < \infty$	$\int_0 \frac{\Phi(u)}{\lvert\Psi(u)\rvert} du = \infty$
$b < 0$	$\eta_t(\lambda) Y_t \xrightarrow{d}$ proper	$\eta_t(\lambda) Y_t \xrightarrow{p} \infty$
$b \geq 0$	$Y_t \xrightarrow{d}$ proper	$Y_t \xrightarrow{p} \infty$

Subcritical and critical CBI

- Consider a special class of CBI processes given by

$$Y_t = Y_0 - \int_0^t (a - bY_s)ds + \int_0^t \sigma\sqrt{Y_s}dB_s + \int_0^t \int_0^{Y_{s-}} \int_0^\infty \xi \tilde{N}(ds, du, d\xi),$$

where $b \geq 0$,

$$\Psi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_0^\infty (e^{-z\xi} - 1 + z\xi)\mu(d\xi).$$

$$\Phi(z) = az.$$

- Y_t is stationary, i.e.,

$$Y_t \xrightarrow{d} Y_\infty, \quad t \to \infty.$$

Synchronous coupling when Grey condition holds

Theorem (Li and M. (2015)). Assume Grey condition holds, i.e.

$$\int_{\theta}^{\infty} \Psi(z)^{-1} dz < \infty.$$

the (sub)critical CBI process with the transition semigroup $(P_t)_{t \geq 0}$ has the strong Feller property. Moreover, for any $t > 0$ and $x, y \in \mathbb{R}_+$, we have

$$\left\| P_t(x, \cdot) - P_t(y, \cdot) \right\|_{var} \leq 2(1 - e^{-\bar{v}_t |x-y|}),$$

which goes to 0 as $t \to \infty$. In this case, the CBI processes have successful coupling.

Synchronous coupling

- Construct the flow $\{Y_t(x) : t \geq 0, x \geq 0\}$ by

$$\begin{aligned} Y_t(x) &= x + \int_0^t (a - bY_s(x))ds + \sigma \int_0^t \int_0^{Y_{s-}(x)} W(ds, du) \\ &+ \int_0^t \int_0^{Y_{s-}(x)} \int_0^\infty \xi \tilde{N}(ds, du, d\xi). \end{aligned}$$

  For fixed x, the solution $\{Y_t(x), t \geq 0\}$ is a CBI process
- Dawson and Li (2012):
  For any $x \geq y \geq 0$ we have $\mathbb{P}(Y_t(x) \geq Y_t(y) \text{ for all } t \geq 0) = 1$ and $(Y_t(x) - Y_t(y))_{t \geq 0}$ is a CB process with branching mechanism Ψ.
- The coupling time is the extinction time of the above CB process.

Exponential ergodicity

- Assume Grey condition holds. Then

  (i) the subcritical CBI process is exponentially ergodic, i.e.

$$\|P_t(x,\cdot) - \mu(\cdot)\|_{var} \leq 2(x\bar{v}_1 + M_\gamma)e^{-\gamma b(t-1)},$$

  where μ is the stationary measure, $\gamma = \delta \wedge 1$ and

$$M_\gamma = \begin{cases} \gamma \bar{v}_1^\gamma \int_0^\infty (1 - L_\mu(\lambda))\lambda^{-(1+\gamma)} d\lambda & \text{if } \gamma < 1, \\ \bar{v}_1 b^{-1}(a + \int_0^\infty u\,\nu(du)) & \text{if } \gamma = 1. \end{cases}$$

  (ii) the critical CBI process is ergodic.

Grey condition is necessary?

- Li, Wang, Li and Zhou (2022): Example of CBI processes, where

$$\mu(d\xi) = 1_{(u,v)}(\xi)d\xi, \quad \text{for some } 0 < u < v < 1.$$

and thus

$$\Psi(z) = \big(b + \frac{v^2 - u^2}{2}\big)z \Rightarrow \int^{\infty} \frac{1}{\Psi(z)}dz = \infty.$$

The CBI-process is exponentially ergodic relative in total variation distance (successful refined basic coupling!)

- Wang (2011): consider Ornstein-Uhlenbeck type processes.

$$X_t(x) = x - \int_0^t bX_s(x)ds + \int_0^t \int_{\mathbb{R}} \xi \tilde{N}_1(ds, d\xi)$$

where $b \geq 0$ and $N_1(ds, d\xi)$ is a Poisson random measure on $(0, \infty) \times \mathbb{R}$ with intensity $ds\nu(d\xi)$

Coupling for OU type processes

Theorem (Wang (2011)). Assume that there exists some $z_0 \in \mathbb{R}$ and some $\varepsilon > 0$ with $B(z_0, \varepsilon) \subset \mathbb{R} \backslash \{0\}$ such that the Lévy measure $\nu(d\xi)$ has an absolutely continuous part in $B(z_0, \varepsilon)$, i.e.,

$$\nu(d\xi) \geq \rho(\xi) d\xi$$

for some non-negative function ρ and

$$\int_{B(z_0, \varepsilon)} \rho(\xi)^{-1} d\xi < \infty.$$

Then for the OU type process

$$\|P_t(x, \cdot) - P_t(y, \cdot)\|_{var} \leq \frac{K(1 + |x - y|)}{\sqrt{t}}, \quad x, y \in \mathbb{R}_+, \ t > 0,$$

for some constant $K > 0$.

Coupling by change of measure

- **Proposition** (Mecke's formula). Let $M(dx)$ be a Poisson random measure on a polish space E with intensity measure $\Lambda(dx)$. Let E_p be the space of point measure on E and $G : E \times E_p \to \mathbb{R}_+$ be some measurable functional. Then

$$\mathbb{E}\Big[\int_E G(x, M)M(dx)\Big] = \int_E \mathbb{E}\big[G(x, M + \delta_x)\big]\Lambda(dx)$$

- Fix $t > 0$. Consider the family of OU type processes by

$$X_t(x) = e^{-bt}x + \int_0^t \int_E \xi e^{-b(t-s)} \tilde{N}_1(ds, d\xi)$$

and

$$X_t(x) - X_t(y) = e^{-bt}(x - y)$$

$$X_t(x) = e^{-bt}x + \int_0^t \int_{B(z_0,\varepsilon/2)} \xi e^{-b(t-s)} N_1(ds, d\xi)$$

and

$$X_t(x) - X_t(y) = e^{-bt}(x - y)$$

Coupling by change of measure

- Let τ be a random variable on $[0, \infty)$ with distribution $\frac{1}{t}1_{[0,t]}(s)ds$ and U with distribution

$$\frac{1_{B(z_0,\varepsilon/2)}(\xi)\rho(\xi)d\xi}{\nu(B(z_0,\varepsilon/2))},$$

  which independent of N_1.
- Add a random point as follows:

$$X_t(x) = e^{-bt}x + \cdots + \int_0^t \int_{B(z_0,\varepsilon/2)} \xi e^{-b(t-s)}\left(N_1 + \delta_{(\tau,U)}\right)(ds, d\xi)$$

$$\begin{aligned} X_t(y) &= e^{-bt}y + \cdots \\ &+ \int_0^t \int_{B(z_0,\varepsilon/2)} \xi e^{-b(t-s)}\left(N_1 + \delta_{(\tau,U+e^{-b\tau}(x-y))}\right)(ds, d\xi) \end{aligned}$$

Subcritical and critical CBI

- Consider a special class of CBI processes given by

$$\begin{aligned} Y_t &= Y_0 - \int_0^t (a - bY_s)ds + \int_0^t \sigma\sqrt{Y_s}dB_s \\ &+ \int_0^t \int_0^{Y_{s-}} \int_0^\infty \xi \tilde{N}(ds, du, d\xi), \end{aligned}$$

where $b \geq 0$,

$$\Psi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_0^\infty (e^{-z\xi} - 1 + z\xi)\mu(d\xi).$$

$$\Phi(z) = az.$$

Coupling for CBI processes (Grey condition possibly fails)

- Assumption A: there exists some $z_0 \in \mathbb{R}_+$ and some $\varepsilon > 0$ with $B(z_0, \varepsilon) \subset (0, \infty)$ such that the Lévy measure $\mu(d\xi)$ has an absolutely continuous part in $B(z_0, \varepsilon)$, i.e.,

$$\mu(d\xi) \geq \rho(\xi) d\xi$$

for some non-negative function ρ and

$$\int_{B(z_0,\varepsilon)} \rho(\xi)^{-1} d\xi < \infty.$$

- Assumption A holds. Then for the subcritical CBI process

$$\|P_t(x,\cdot) - P_t(y,\cdot)\|_{var} \leq \frac{K(1 + |x-y|)}{\sqrt{t}}, \quad x, y \in \mathbb{R}_+, \ t > 0,$$

for some constant $K > 0$.

Idea of proof

- Step 1: construct the flow $\{Y_t(x) : t \geq 0, x \geq 0\}$ by

$$\begin{aligned} Y_t(x) &= x + \int_0^t (a - bY_s(x))ds + \sigma \int_0^t \int_0^{Y_{s-}(x)} W(ds, du) \\ &\quad + \int_0^t \int_0^{Y_{s-}(x)} \int_0^\infty \xi \tilde{N}(ds, du, d\xi). \end{aligned}$$

  and

$$Y_t(x) = Y_t(x) - Y_t(0) + Y_t(0)$$

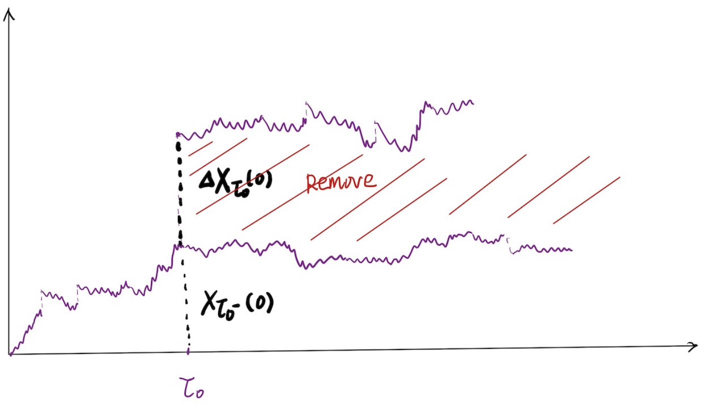
  where $Y_t(x) - Y_t(0)$ is a (sub)critical CB process independent of $Y_t(0)$.

- Step 2: Let

$$\tau_0 = \inf\{t > 0 : \Delta Y_t(0) \in B_{\varepsilon/2}\}$$

  where $B_{\varepsilon/2} = B(z_0, \varepsilon/2)$.

Remove the CB process starting from $\Delta Y_{\tau_0}(0)$ for $t \in [\tau_0, \infty)$



The remaining CBI process is given by

$$\hat{Y}_t(0) = \int_0^t (a - \hat{b}\hat{Y}_s(0))ds + \sigma \int_0^t \int_0^{\hat{Y}_{s-}(0)} W(ds, du) + \int_0^t \int_0^{\hat{Y}_{s-}(0)} \int_{B^c_{\varepsilon/2}} \xi \tilde{N}(ds, dz, d\xi).$$

The key decomposition for CBI processes

- $D[0,\infty)$: the space of càdlàg paths $t \mapsto w_t$ from $[0,\infty)$ to $\mathbb{R}_+$.
- $\mathbb{Q}_x(dw)$ denote the distribution on $D[0,\infty)$ of the CB process $(X_t(x) : t \geq 0)$ with $X_0(x) = x$.
- $\mathbb{Q}_\mu(dw)$ on $D[0,\infty)$ by

$$\mathbb{Q}_\mu(dw) = \int_{B_{\varepsilon/2}} \mu(dx)\mathbb{Q}_x(dw).$$

- $M(ds, du, dw)$ be a Poisson random measure on $(0,\infty)^2 \times D[0,\infty)$ with intensity measure $dsdu\mathbb{Q}_\mu(dw)$.

We define the process $Y_t(x)$ by

$$Y_t(x) = Y_t(x) - Y_t(0) + \hat{Y}_t(0) + \int_0^t \int_0^{\hat{Y}_{s-}(0)} \int_{D[0,\infty)} w_{t-s} M(ds, du, dw)$$

Note that $\{Y_t(x) - Y_t(0)\}$, $\{\hat{Y}_t(0)\}$ and $M(ds, du, dw)$ are independent of each other

Coupling by change of conditional measure

- Let $\hat{\mathbb{P}}(\cdot) = \mathbb{P}(\cdot|\hat{Y}_s(0), 0 \le s \le t)$
- Under $\hat{\mathbb{P}}$, let τ be a random variable on $[0,\infty)$ with distribution $\frac{1}{\int_0^t \hat{Y}_s(0)ds} 1_{[0,t]}(s)\hat{Y}_s(0)ds$ and ω on $D[0,\infty)$ with distribution

$$\frac{1_{B_{\varepsilon/2}}(w_0)\mathbb{Q}_\mu(dw)}{\mu(B_{\varepsilon/2})},$$

  which independent of M.
- Add a random point which induces a independent CB path as follows:

$$Y_t(x) = \cdots + \int_0^t \int_0^{\hat{Y}_{s-}(0)} \int_{D[0,\infty)} w_{t-s}\big(M + \delta_{(\tau,\omega)}\big)(ds, du, dw),$$

$$Y_t(y) = \cdots + \int_0^t \int_0^{\hat{Y}_{s-}(0)} \int_{D[0,\infty)} w_{t-s}\big(M + \delta_{(\tau,\omega+Y_{\tau+\cdot}(y)-Y_{\tau+\cdot}(x))}\big)(ds,$$

Thanks for your attention !